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AND THE CAUCHY - RIEMANN EQUATIONS

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A UNIFIED NUMERICAL TREATMENT OF THE WAVE EQUATION AND THE CAUCHY - RIEMANN EQUATIONS

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ABSTRACT

A unified second order accurate finite difference approach to these problems is discussed and a motivation as a weak solution method is described.

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I. Introduction

Consider a rectangular domain D in the x-y plane in which a vector solution to the differential equation

$$U_y + AU_x = 0 \tag{1.1}$$

is to be obtained when certain conditions, described by a boundary operator B_A, are prescribed on the boundary γ of D, viz., $B_AU = g$ on γ. We assume B_A is such that a well posed problem results.

The wave equation results when $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in which case rank $B_A = 2$ on the side where initial conditions are prescribed, rank $B_A = 0$ on the opposite side, and rank $B_A = 1$ on the remaining sides. We recall that if

$$||U||_y^2 = \int_{x_0}^{x_1} U^T(x,y)\, U\,(x,y)\,dx$$

the familiar energy estimate

$$\frac{1}{2}\frac{d}{dy}\, ||U||_y^2 + U^T(x,y)\,AU\,(x,y)\Big|_{x_0}^{x_1} = 0 \tag{1.2}$$

results by multiplying (1.1) by $U^T(x,y)$ and integrating. If the boundary conditions are dissipative, $U^TAU|_{x_1} \geq U^TAU|_{x_0}$ so that $||U||_y \leq ||U||_0$; as a result both existence and uniqueness can be shown to follow.

For the Cauchy - Riemann equations $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and rank $B_A = 1$ on all sides. In this case an estimate for the norm

$$||U||^2 = \iint_D U^TUdxdy$$

in terms of boundary data can be given by introducing a potential function and employing Green's theorem; again, uniqueness as well as existence under rather general conditions follows.

Friedrich's theory of symmetric positive equations [1] provides a unified viewpoint for treating systems of equations without regard to type and it is natural to inquire whether or not a more unified viewpoint about numerical approximations is also possible.

This paper investigates a finite difference scheme for (1.1) which is convergent for the Cauchy-Riemann equations as well as for the wave equation. For the latter a direct parallel to the energy argument described above is obtained. For the Cauchy-Riemann equations, a simple argument will imply the maximum principle and its well known consequences.

A motivation for the finite difference scheme as the consequence of a common approximation method for both classes of problems is given in the final section.

In the following sections we assume that the rectangular domain D is divided into MN rectangular cells π_{ij} centered at points P_{ij} and each of area $\Delta x \Delta y$; $U_{ij} \equiv U(P_{ij})$. The finite difference scheme we shall be primarily concerned with involves values at the midpoints of the sides of π_{ij} as indicated in Figure 1.

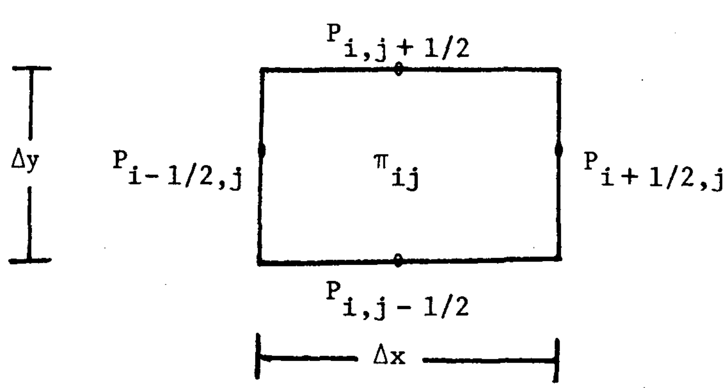

Figure 1: An elementary cell π_{ij}

The finite difference equations associated with equation (1.1) in each cell π_{ij} are:

$$\text{(1.3)} \quad \begin{aligned} &\text{(a)} \quad (U_{i,j+1/2} - U_{i,j-1/2}) + \lambda A\,(U_{i+1/2,j} - U_{i-1/2,j}) = 0 \\ &\text{(b)} \quad U_{i,j+1/2} + U_{i,j-1/2} = U_{i+1/2,j} + U_{i-1/2,j} \end{aligned}$$

in which $\lambda = \Delta y/\Delta x$. The consistency of these approximations to (1.1) is evident.

Note that in D (1.3) express $2NM$ equations for the $(N-1)M+(M-1)N$ unknowns at points interior to D and the $2(N+M)$ unknowns lying on the boundary of D; the additional $(N+M)$ boundary conditions for each problem (arising from the conditions for rank B_A) thus result in a determined system of algebraic equations.

II. The Wave Equation

In the case of the wave equation A is symmetric and an energy estimate analogous to (1.2) results by multiplying the difference equation (1.3a) by $(U_{i,j+1/2} + U_{i,j-1/2})^T$ and summing on i; interchanging the average in the y direction with the average in the x direction as expressed by (1.3b) and employing the symmetry of A there results, with

$$\|U\|^2_{j+1/2} = \sum_i U^T_{i,j+1/2}\, U_{i,j+1/2} ,$$

$$\|U\|^2_{j+1/2} = \|U\|^2_{j-1/2} - \lambda U^T_{i,j}\, AU_{i,j}\Big|_{i=0}^{i=M} . \tag{2.1}$$

For dissipative boundary conditions appropriate to the wave equation the last term on the right-hand side is non-negative and the norm estimate

$$||U||^2_{j+1/2} \leq ||U||^2_{j-1/2}$$

results, a fact which enables one to conclude convergence for all values of λ as $\Delta x \to 0$, $\Delta y \to 0$ when solutions are smooth.

This scheme is non-dissipative in the interior of D. As will be indicated below, (1.3) is accurate to second order and well-known results about non-dissipative second order methods indicate that the scheme cannot provide a monotonic approximation in a neighborhood of a discontinuity of the solution of (1.1). Several means of overcoming this problem are known and will not be discussed further in this paper.

In (1.3) either $U_{i,j+1/2}$ or $U_{i,j-1/2}$ may be eliminated; the result is

$$\text{(2.2)}\quad \begin{aligned} &\text{(a)} \quad R^+U_{i+1/2,j} + R^-U_{i-1/2,j} = U_{i,j-1/2} \ , \\ &\text{(b)} \quad R^-U_{i-1/2,j} + R^+U_{i-1/2,j} = U_{i,j+1/2} \end{aligned}$$

where

$$R^{\pm} = \frac{1}{2}(I \pm \lambda A) \quad .$$

Thus, with $U_{i,j-1/2}$ known for a fixed value of j, (2.2a) describes a two-point boundary value problem to be solved for $U_{i,j}$; with $U_{i,j}$ determined, (2.2b) then provides values $U_{i,j+1/2}$ as new initial data for the next step. More directly, by eliminating the values $U_{i,j+1/2}$ between two neighboring cells the following box-scheme results:

$$\text{(2.3)}\quad \begin{aligned} &R^+U_{i+1/2,\,j+1} + R^-U_{i-1/2,\,j+1} \\ &\quad = R^-U_{i+1/2,j} + R^+U_{i-1/2,j} \end{aligned}$$

and from which the assertion that the proposed scheme (1.3) is second-order accurate is immediately evident.

This discussion can be extended to hyperbolic systems of conservation laws of the form $U_t + F_x + G_y = 0$ and for which the approximation method underlying (1.3) which is described below also indicates a natural operator splitting technique for resolving the solution as a simple composition of one-dimensional solutions each of which is described by (1.3). This, and other aspects of this problem, will be discussed in detail in a separate paper.

III. Cauchy - Riemann Equations

We shall limit our discussion of the problem for the Cauchy - Riemann equations to that in which one component of U is prescribed on the boundary of D, i.e., to the Dirichlet problem for (1.1) when $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In order to discuss equation (1.3) in this case it will be convenient to introduce new variables $W(Q)$ associated with vertex points of a cell π_{ij} as indicated in Figure 2

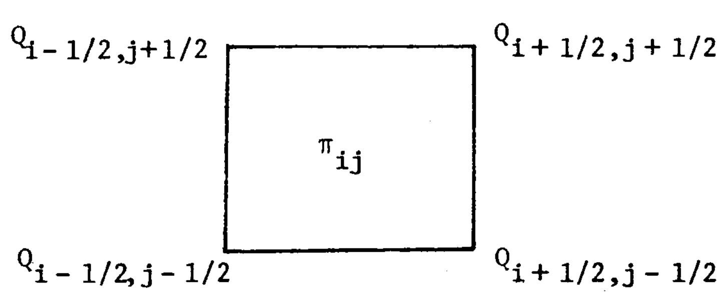

Figure 2

satisfying

$$2U_{i\pm 1/2,j} = W(Q_{i\pm 1/2,j+1/2}) + W(Q_{i\pm 1/2,j-1/2}) ,$$

(3.1)

$$2U_{i,j\pm 1/2} = W(Q_{i+1/2,j\pm 1/2}) + W(Q_{i-1/2,j\pm 1/2}) .$$

in which case equation (1.3b) is satisfied identically while (1.3a) results in the box-scheme

$$R^{+}(W_{i+1/2,j+1/2} - W_{i-1/2,j-1/2})$$

(3.2)

$$+R^{-}(W_{i-1/2,j+1/2} - W_{i+1/2,j-1/2}) = 0 .$$

To (3.1), (3.2) are to be added $(N+M)$ boundary the conditions $B_A U = g$ now expressed in terms of values of W.

The number of unknown vectors $W(Q)$ occurring in these equations is $(M+1)(N+1)$. If $W(Q)$ is arbitrarily specified at any point then equations (3.2) and the $(M+N)$ boundary conditions will yield a determined system of equations for $W(Q)$.

The fact that $W(Q)$ may be arbitrarily specified at any point reflects the fact that the solution of (3.1) is determined only to within an arbitrary constant vector.

Consider the vertex point Q_0 common to the four adjacent cells π_{ij}, $\pi_{i+1,j}$, $\pi_{i,j+1}$, $\pi_{i+1,j+1}$ indicated in Figure 3.

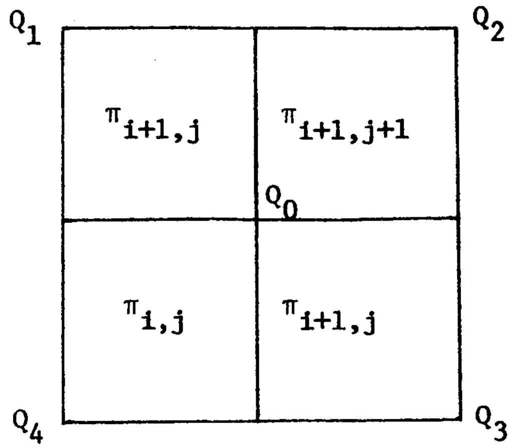

Figure 3

Using (3.2) the reader may easily verify that

$$4W(Q_0) = W(Q_1) + W(Q_2) + W(Q_3) + W(Q_4) \ . \tag{3.3}$$

This represents an approximation to the Laplacian operator rotated through an angle of $\pi/4$ about Q_0.

This observation implies a maximum principle for $W(Q)$, hence uniqueness for $U(P)$; familiar related arguments also establish the (quadratic) convergence of $U(P)$ to the solution of the Cauchy-Riemann equations as $\Delta x \to 0$, $\Delta y \to 0$. Because of their familiarity they are not repeated here.

A consequence of (3.3), also, is that the semi-norm

$$\|W\|^2 = \sum_{Q} \sum_{Q' \varepsilon N(Q)} (W(Q) - W(Q')^T (W(Q) - W(Q')) \ ,$$

where $N(Q)$ indicates the diagonal neighbor of Q in a cell, provides an estimate for W in terms of the boundary data, using the analogue of Green's theorem.

Equation (3.3) expressed in terms of box variables, originally introduced to facilitate a proof of the convergence of (1.3), also suggests effective solution techniques for this problem based upon known results.

IV. A Motivation

Consider the possibility of developing an approximation method which applies to the two completely different types of equations expressed by (1.1). Since the wave equation permits discontinuous solutions it is natural to consider a method which is consistent with weak-solution methods. When applied to elliptic equations such mesh approximation methods should, as well, be expected to converge to the known strong solution of such problems.

To this end consider the following adaptation of the weak-element approximation method described in [2]: let σ denote a side of one of the elementary rectangular cells π which partition D and let $\pi(\sigma)$ be an associated cell of the same dimensions bisected by the side σ.

In each elementary cell π with the origin as center let

$$U(x,y,\pi) = \underline{a}(\pi) + (x - yA)\,\underline{b}(\pi) \tag{4.2}$$

where $\underline{a}(\pi)$, $\underline{b}(\pi)$ are parameters yet to be determined; thus $U(x,y,\pi)$ is a solution of (3.1) in π.

On each associated cell $\pi(\sigma)$ let $\Phi(\sigma)$ be smooth and equal to 1 in $\pi(\sigma)$ except in a small neighborhood of the boundary where it vanishes. Considering $\Phi = \{\Phi(\sigma)\}$ as a class of test functions, multiply (1.1) by $\Phi(\sigma)$ and integrate by parts over a subdomain $\hat{\pi}(\sigma)$ of $\pi(\sigma)$ to obtain

$$0 = \int_{\partial\hat{\pi}(\sigma)} \Phi(\sigma)(dx - dyA)U - \iint_{\hat{\pi}(\sigma)} (\Phi_y(\sigma) - \Phi_x(\sigma)A)Udxdy \; . \tag{4.3}$$

Thus if σ is a side common to cells π, π' then the approximations $U(P,\pi)$, $U(P,\pi')$ will satisfy (4.3) if

$$\int_\sigma U(P,\pi)d\sigma = \int_\sigma U(P,\pi')d\sigma \tag{4.4a}$$

while at a boundary side $\bar{\sigma}$

$$\int_{\bar{\sigma}} (B_A U(P,\pi) - g)d\bar{\sigma} = 0 \; . \tag{4.4b}$$

In the examples under discussion these conditions provide for a determined algebraic system of equations for the parameters $\underline{a}(\pi)$, $\underline{b}(\pi)$.

For the linear approximation described by (4.2), equation (4.3) implies that the values of the approximation $U(P,\pi)$ itself is continuous at the center point of each side σ of π . As a result, the values of the parameters $\underline{a}(\pi)$, $\underline{b}(\pi)$ occurring in (4.2) may be expressed in terms of values of U at the center points of the sides of π . The results are the difference equations (1.3).

The previous discussion shows that this approximation method produces convergent approximations for all values of the mesh parameters for both the wave equation as well as the Cauchy - Riemann equations. More generally, this suggests that a more unified approach can be developed to treat the numerical solution of differential equations without regard to type.

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REFERENCES

[1] K. O. FRIEDRICHS, *Symmetric positive linear differential equations*, Comm. Pure Appl. Math., Vol. II (1958), pp. 333-418.

[2] M. E. ROSE, *Weak-element approximations to elliptic differential equations*, Numer. Math., 24 (1975), pp. 185-204.